Discrete final-offer arbitration model (Development of the optimization theory for the dynamic systems and their applications)

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Discrete final-offer arbitration model

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Abstract

A bargaining problem with two players Labor (player L) and Management (player M) is considered. The players must decide the monthly wage payed to L by M. At the begining players L and M submit their offers s_1 and s_2. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator is called in and he chooses the offer which is nearest for his solution α. We suppose that a solution α is concentrated in two points $a, 1-a$ at the interval $[0,1]$ with probabilities $p, q = 1-p$. The equilibrium in the arbitration game among pure and mixed strategies is derived.

Key words: bargaining problem, arbitration, equilibrium strategy.

AMS Subject Classification: 91A05,91A80, 91B26.

1 Introduction

We consider a zero-sum game related with a model of the labor-management negotiations using an arbitration procedure. Imagine that two players: Labor (player L) and Management (player M) bargain on a wage bill which has to be in the range $[0,1]$ where the current wage bill is normalised at zero, and the known maximum management ability to pay is at 1. Player L is interested to maximize a wage bill as much as possible and the player M has the opposite goal.

At the begining the players L and M submit their offers s_1 and s_2 respectively, $s_1, s_2 \in [0,1]$. If $s_1 \leq s_2$ there is an agreement at $(s_1 + s_2)/2$. If not, the arbitrator A is called in and he has to choose one of the decisions.

There are different approaches in analyzing the arbitration models [1-6]. We consider here the final-offer arbitration procedure [3] which allows the arbitrator only to choose one of the two final offers made by the players. We suppose here that the arbitrator imposes a solution α which is random variable being concentrated in two points a and $b = 1-a$ with different probabilities p and $q = 1-p$, $0 \leq a, p \leq 1$. The arbitrator chooses the offer which is nearest for his solution α. The solution of this game with equal $p = q = 1/2$ was obtained in [6]. In this paper we obtain the solution of this game where p and q can be non-equal.

So, we have a zero-sum game determined in the unit square where the strategies of players L and M are the real numbers $s_1, s_2 \in [0,1]$ and payoff function in this game has form $H(s_1,s_2) = EH_\alpha(s_1,s_2)$, where

$$H_\alpha(s_1,s_2) = \begin{cases} (s_1+s_2)/2, & if \quad s_1 \le s_2 \\ s_1, & if \quad s_1 > s_2, |s_1-\alpha| < |s_2-\alpha| \\ s_2, & if \quad s_1 > s_2, |s_1-\alpha| > |s_2-\alpha| \\ \alpha, & if \quad s_1 > s_2, |s_1-\alpha| = |s_2-\alpha| \end{cases} \tag{1}$$

Below we show that the equilibrium in this game in dependence on value a can be among pure (section 2) and mixed (sections 3-4) strategies.

2 Solution of the game. Pure strategies

Theorem 1. Let $p \in (0, 0.5]$ and $a \in [0, p/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$. The value of the game $v = q$.

Proof. Let player II uses $s_2 = 0$. The payoff of player I is equal to:
for $s_1 \in [0, 2a)$ $H(s_1, 0) = ps_1 + qs_1 = s_1 < 2a \le p \le q$,
for $s_1 = 2a$ $H(2a, 0) = pa + (1-p)2a = (2-p)a < 2a \le p \le q$,
for $s_1 \in (2a, 1]$ $H(s_1, 0) = p0 + qs_1 = qs_1$.

The maximum of the function is reached for $s_1 = 1$ and equals to q. Now, suppose that player I uses $s_1 = 1$. For $s_2 \in [0, 1-2a)$ $H(1, s2) = ps_2 + q$. Minimum of this function lies in $s_2 = 0$ and equal to q. For $s_2 = 1-2a$ $H(1, 1-2a) = p(1-2a) + q(1-a) = 1-a-ap$. Because $p/(1+p) > p/2 \ge a$, it follows $p > a + ap$ and $1-a-ap > 1-p = q$. For $s_2 \in (1-2a, 1]$ $H(s_1, s_2) = ps_2 + qs_2 = s_2$. According to condition $p \ge 2a$ we have $s_2 \ge 1-2a > 1-p = q$. So, for all s_2 $H(1, s_2) \ge q$ and $H(s_1, 0) \le q$ for all s_1. Hence, $\{s_1 = 1, s_2 = 0\}$ ia an equiliblium in the game and $v = q$.

Analogous arguments leads to

Theorem 2. Let $p \in (0.5, 1)$ and $a \in [0, q/2]$. Equilibrium consists of pure strategies and has form $s_1^* = 1, s_2^* = 0$, and value of the game $v = q$.

3 Method for obtaining the equilibrium among mixed startegies

In case $a > \min\{p/2, q/2\}$ equilibrium consists of mixed strategies, i.e. randomised strategies of players L and M. Denote $F_1(s_1)$ and $F_2(s_2)$ distribution functions of the strategies for L and M, respectively. Suppose, that $F_1(s_1)$ $\left[F_2(s_2)\right]$ is continuous and its support consists of two intervals $(\alpha_1; \alpha_2]$ and $(\alpha_3; \alpha_4]$ $\left[(\beta_1; \beta_2], (\beta_3; \beta_4]\right]$ at the $[0; 1]$ with $\alpha_2 \le \alpha_3$ $\left[\beta_2 \le \beta_3\right]$.

In extreme points of the interval $[0;1]$ functions $F_1(s_1)$ and $F_2(s_2)$ can have a gap. Let also $\beta_4 \le \alpha_1$, $F_1(\alpha_1) = 0$ and $F_2(\beta_4) = 1$.

Let $F_{1,12}(s_1)$ and $F_{1,34}(s_1)$ denote the form of $F_1(s_1)$ at the intervals $(\alpha_1;\alpha_2]$ and $(\alpha_3;\alpha_4]$; and, respectively, $F_{2,12}(s_2)$ and $F_{2,34}(s_2)$ – for the function $F_2(s_2)$ at $(\beta_1;\beta_2]$ and $(\beta_3;\beta_4]$.

Firtsly, consider the case $p \le 0.5$. Admit, that the intervals $(\alpha_1;\alpha_2]$ and $(\beta_1;\beta_2]$ are symmetric in respect on the point a and the intervals $(\alpha_3;\alpha_4]$ and $(\beta_3;\beta_4]$ are symmetric in respect on b. Otherwords,

$$\alpha_1 = 2a - \beta_2, \quad \beta_1 = 2a - \alpha_2, \quad \alpha_4 = 2b - \beta_3, \quad \beta_4 = 2b - \alpha_3. \tag{2}$$

Suppose, that player L (M) uses a mixed strategy $F_1(s_1)$ $\Big(F_2(s_2)\Big)$ and consider the payoffs of the players.

For $s_1 \in (\alpha_1;\alpha_2]$,

$$H(s_1, F_2(s_2)) = p\left\{ s_1 F_{2,12}(2a - s_1) + \int_{2a-s_1}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\} + q s_1. \tag{3}$$

For $s_1 \in (\alpha_3;\alpha_4]$,

$$H(s_1, F_2(s_2)) = p\left\{ 0 \cdot F_2(0) + \int_{2a-\alpha_2}^{\beta_2} s_2 dF_{2,12}(s_2) + \int_{\beta_3}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\}$$

$$+q\left\{ s_1 F_{2,34}(2b - s_1) + \int_{2b-s_1}^{2b-\alpha_3} s_2 dF_{2,34}(s_2) \right\}. \tag{4}$$

For $s_2 \in (\beta_1;\beta_2]$,

$$H(F_1(s_1), s_2) = p\left\{ \int_{2a-\beta_2}^{2a-s_2} s_1 dF_{1,12}(s_1) + s_2(1 - F_{1,12}(2a - s_2)) \right\}$$

$$+q\left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \int_{\alpha_3}^{2b-\beta_3} s_1 dF_{1,34}(s_1) + 1 \cdot (1 - F_1(1)) \right\}. \tag{5}$$

For $s_2 \in (\beta_3;\beta_4]$,

$$H(F_1(s_1), s_2) = p s_2 + q\left\{ \int_{2a-\beta_2}^{\alpha_2} s_1 dF_{1,12}(s_1) + \right.$$

$$\left. + \int_{\alpha_3}^{2b-s_2} s_1 dF_{1,34}(s_1) + s_2(1 - F_{1,34}(2b - s_2)) \right\}. \tag{6}$$

If $F_1^*(s_1)$, $F_2^*(s_2)$ are optimal then the equations $H(s_1, F_2^*(s_2)) = v$ and $H(F_1^*(s_1), s_2) = v$, must be satisfied in the support-intervals where v-value of the game. Hence,

$$H(s_1, F_2^*(s_2)) = v, \quad s_1 \in (\alpha_1;\alpha_2] \cup (\alpha_3;\alpha_4],$$

$$H(F_1^*(s_1), s_2) = v, \quad s_2 \in (\beta_1;\beta_2] \cup (\beta_3;\beta_4].$$

From here,

$$\frac{\partial H(s_1, F_2^*(s_2))}{\partial s_1} = 0, \quad s_1 \in (\alpha_1;\alpha_2] \cup (\alpha_3;\alpha_4],$$
$$\frac{\partial H(F_1^*(s_1), s_2)}{\partial s_2} = 0, \quad s_2 \in (\beta_1;\beta_2] \cup (\beta_3;\beta_4].$$

Finding the derivative of (3-4) in s_1 and putting it equal to 0, and using the admission that $F_2^*(\beta_4) = 1$ and $F_2^*(s_2)$ is continuous at $[\beta_2;\beta_3]$, consequently, $F_2^*(\beta_2) = F_2^*(\beta_3)$, we obtain the system of differential equations with boundary conditions:

$$\begin{array}{l} p\left\{2(s_1 - a)F_{2,12}^{*'}(2a - s_1) - F_{2,12}^*(2a - s_1)\right\} - q = 0, \quad s_1 \in (\alpha_1;\alpha_2], \\ q\left\{2(b - s_1)F_{2,34}^{*'}(2b - s_1) + F_{2,34}^*(2b - s_1)\right\} = 0, \quad s_1 \in (\alpha_3;\alpha_4], \\ F_{2,34}^*(\beta_4) = 1, \quad F_{2,12}^*(\beta_2) = F_{2,34}^*(\beta_3). \end{array}$$

Changing the arguments $t_1 = 2a - s_1$, $t_1 \in (\beta_1;\beta_2]$ in the first equation and $t_2 = 2b - s_1$, $t_2 \in (\beta_3;\beta_4]$ in the second one we obtain the system:

$$\frac{dt_1}{2(a - t_1)} = \frac{dF_{2,12}^*}{F_{2,12}^* + p/q}, \quad \frac{dt_2}{2(b - t_2)} = \frac{dF_{2,34}^*}{F_{2,34}^*}.$$

The solution which satisfies the boundary conditions has the following form

$$F_2^*(s_2) = \begin{cases} 0, & \text{if } s_2 \le 2a - \alpha_2, \\ \left(\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + \frac{q}{p}\right)\frac{\sqrt{a - \beta_2}}{\sqrt{a - s_2}} - \frac{q}{p}, & \text{if } 2a - \alpha_2 < s_2 \le \beta_2, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}, & \text{if } \beta_2 < s_2 \le \beta_3, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - s_2}}, & \text{if } \beta_3 < s_2 \le 2b - \alpha_3, \\ 1, & \text{if } 2b - \alpha_3 < s_2. \end{cases} \tag{7}$$

Finding the derivative of (5-6) in s_2 and putting it equal to 0, and using the admission $F_1^*(\alpha_1) = 0$ and $F_1^*(\alpha_2) = F_1^*(\alpha_3)$, we obtain the system:

$$\begin{array}{l} p\left\{1 - F_{1,12}^*(2a - s_2) - 2(a - s_2)F_{1,12}^{*'}(2a - s_2)\right\} = 0, \quad s_2 \in (\beta_1;\beta_2], \\ p + q\left\{1 - F_{1,34}^*(2b - s_2) - 2(b - s_2)F_{1,34}^{*'}(2b - s_2) = 0\right\}, \quad s_2 \in (\beta_3;\beta_4], \\ F_{1,12}^*(\alpha_1) = 0, \quad F_{1,12}^*(\alpha_2) = F_{1,34}^*(\alpha_3). \end{array}$$

Let change the arguments $t_1 = 2a - s_2$, $t_1 \in (\alpha_1;\alpha_2]$ in the first equation, and $t_2 = 2b - s_2$, $t_2 \in (\alpha_3;\alpha_4]$ in the second equation:

$$\frac{dt_1}{2(t_1 - a)} = \frac{dF_{1,12}^*}{1 - F_{1,12}^*}, \quad \frac{dt_2}{2(t_2 - b)} = \frac{dF_{1,34}^*}{1 + p/q + F_{2,34}^*}.$$

The solution of the system:

$$F_1^*(s_1) = \begin{cases} 0, & \text{if } s_1 \le 2a - \beta_2, \\ 1 - \frac{\sqrt{a - \beta_2}}{\sqrt{s_1 - a}}, & \text{if } 2a - \beta_2 < s_1 \le \alpha_2, \\ 1 - \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}, & \text{if } \alpha_2 < s_1 \le \alpha_3, \\ 1 + \frac{p}{q} - \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q}\right)\frac{\sqrt{\alpha_3 - b}}{\sqrt{s_1 - b}}, & \text{if } \alpha_3 < s_1 \le 2b - \beta_3, \\ 1, & \text{if } 2b - \beta_3 < s_1. \end{cases} \tag{8}$$

Now let us substitute the functions (7)–(8) to (3) - (6). For $s_1 \in (\alpha_1;\alpha_2]$,

$$H_1 = H(s_1, F_2^*(s_2)) = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}((2a-\beta_2)-(2b-\beta_3)) + p\alpha_3 + q(2a-\beta_2).$$

For $s_1 \in (\alpha_3;\alpha_4]$,

$$H_2 = H(s_1, F_2^*(s_2)) = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}((2a-\beta_2)-(2b-\beta_3)) + p\alpha_3 + q(2a-\beta_2)-$$

$$-p\alpha_2\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}\cdot\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3.$$

For $s_2 \in (\beta_1;\beta_2]$,

$$H_3 = H(F_1^*(s_1), s_2) = q\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a)-(\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b)-$$

$$-q\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}\cdot\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}} + p\beta_2 + q\theta$$

For $s_2 \in (\beta_3;\beta_4]$,

$$H_4 = H(F_1^*(s_1), s_2) = q\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_2 - 2a)-(\alpha_3 - 2b)) + q\beta_2 - p(\alpha_3 - 2b),$$

where

$$\theta = \begin{cases} 0, & \text{if } F_1^*(1) = 1, \\ -\frac{p}{q} + \left(\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q}\right)\frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } F_1^*(1) < 1. \end{cases}$$

So, take place

$$H_2 = H_1 + \chi_1,$$

$$H_3 = H_4 + \chi_2,$$

where

$$\chi_1 = -p\alpha_2\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}\cdot\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} - q\alpha_2\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} + q\alpha_3,$$

$$\chi_2 = -q\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}\cdot\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2 - a}} - p\beta_3\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}} + p\beta_2 + q\theta.$$

But must be $H_1 = H_2 = v$ and $H_3 = H_4 = v$, hence

$$\chi_1 = 0,$$

$$\chi_2 = 0,$$

$$H_1 = H_4.$$

Below we will find a solution of the system (9) in different cases. The value of the
equal

$$v = p\frac{\sqrt{\alpha_3 - b}}{\sqrt{b-\beta_3}}\cdot\Big((2a-\beta_2)-(2b-\beta_3)\Big) + p\alpha_3 + q(2a-\beta_2).$$

Denote $\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}=\frac{1}{x}$, $\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}=y$. After symplifications (9) can be rewritten:

$$
\begin{gathered}
-\frac{\alpha_2}{x}(py+q)+q\alpha_3=0,\\
-\beta_3 y(q/x+p)+p\beta_2+q\theta=0,\\
p(y(2a-2b-\beta_2+\beta_3)+2\alpha_3-2b)=q\left(\frac{1}{x}(\alpha_2-\alpha_3-2a+2b)+2\beta_2-2a\right).
\end{gathered}
\tag{11}
$$

If $F_1^*(1)=1$ (or, $F_{1,34}^*(2b-\beta_3)=1$, or $\theta=0$), then $y\Big(q/x+p\Big)=p$. Substituting it to (11) we receive $\beta_2=\beta_3$. If $F_1^*(1)<1$ ($2b-\beta_3=1$), then $\beta_3=2b-1$, $y=\frac{\sqrt{\alpha_3-b}}{\sqrt{a}}$ and $q\theta=-p+(q/x+p)y$.

Analogously, if $F_2^*(0+)=0$ ($F_{2,12}^*(2a-\alpha_2)=0$), then $1/x(py+q)=q$. Substituting to (11), we receive $\alpha_2=\alpha_3$. If $F_2^*(0+)>0$ ($2a-\alpha_2=0$), then $\alpha_2=2a$ and $1/x=\frac{\sqrt{a-\beta_2}}{\sqrt{a}}$. Thus, take place $F_1^*(1)=1\Longrightarrow\beta_2=\beta_3$ and $F_2^*(0+)=0\Longrightarrow\alpha_2=\alpha_3$.

Varying different collections of the values $F_1^*(1)$ and $F_2^*(0+)$ and demanding that the support of optimal strategies belongs to $[0;1]$, we will obtain the form of optimal strategies depending on values of a and p (see Fig. 1).

4 Solution of the game. Mixed Strategies

4.1 Equilibrium for $(p,a)\in D_1$

Suppose that $F_1^*(1)=1$ and $F_2^*(0+)=0$ (i.e. $\alpha_2=\alpha_3=A$, $\beta_2=\beta_3$=B). From the equations $\frac{1}{x}=\frac{\sqrt{a-B}}{\sqrt{A-a}}$, $y=\frac{\sqrt{A-b}}{\sqrt{b-B}}$ it follows

$$
\alpha_2=\alpha_3=A=\frac{bx^2(1+y^2)-ay^2(1+x^2)}{x^2-y^2},\qquad \beta_2=\beta_3=B=\frac{a(1+x^2)-b(1+y^2)}{x^2-y^2}. \tag{12}
$$

The first two equations in (11) give

$$
\begin{cases} qx=py+q,\\ y\left(\frac{q}{x}+p\right)=p, \end{cases}
$$

which positive solution is

$$
x=\frac{p^2+pq-q^2+\sqrt{p^4+2p^3q-p^2q^2+2pq^3+q^4}}{2pq}, \tag{13}
$$

$$
y=\frac{p^2-pq-q^2+\sqrt{p^4+2p^3q-p^2q^2+2pq^3+q^4}}{2p^2}. \tag{14}
$$

It is not difficult to check that it satisfies to the third equation in (11).

The values x,y and (12) give the solution of the game iff the following system of inequalities be satisfied

$$\beta_1\geq 0,\quad \alpha_4\leq 1,$$

or

$$a \geq \frac{1+y^2}{3+2y^2-x^2y^2}, \quad a \geq \frac{1+x^2}{3+2x^2-x^2y^2}.$$

The solution of this system is the inequality $a \geq \frac{1+x^2}{3+2x^2-x^2y^2}$ ($\alpha_4 \leq 1$). It determines some region on the plane (p,a), denote it D_1 (see. Fig.1) with the lower border $a_1(p) = \frac{1+x^2}{3+2x^2-x^2y^2}$.

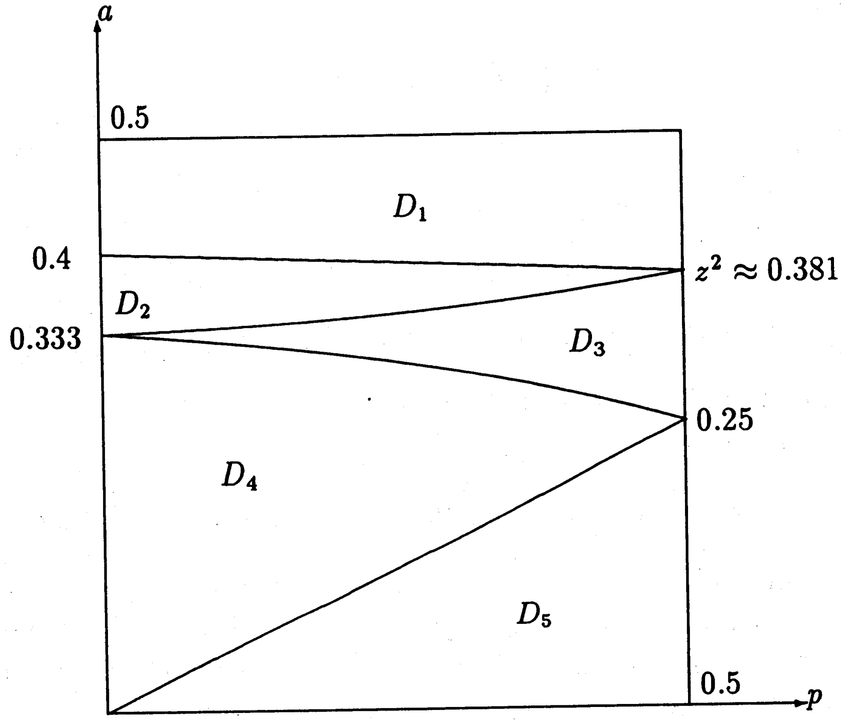

Fig. 1

Theorem 3. For $(p,a) \in D_1$ the equilibrium is (F_1^*, F_2^*) of the form (7–8) with parameters detemined by (12–14). The value of the game : $v = q(2a-\beta_2) + p\alpha_3 - 2p(2b-1)\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_2}}$.

Notice some properties of the solution:

$$\lim_{p\to 0} a_1(p) = 0.4, \qquad \lim_{p\to 0.5} a_1(p) = z^2,$$

where z is the "golden section" of the interval $[0,1]$. It follows from

$$\lim_{p\to 0+} x = 1, \quad \lim_{p\to 0+} y = 0, \quad \lim_{p\to 0.5-} x = \frac{\sqrt{5}+1}{2}, \quad \lim_{p\to 0.5-} y = z = \frac{\sqrt{5}-1}{2}.$$

Notice also, that for fixed p if a decreases then α_4 increases to 1 and reaches it for $a = a_1(p)$ (to obtain it we can substitute $a_1(p)$ instead of a to $\alpha_4 = 2 - 2a - \beta_3$). For values $a \leq a_1(p)$, the solution of the game is different.

4.2 Equilibrium for $(p,a) \in D_2$

If $F_1^*(1) < 1$ and $F_2^*(0+) = 0$ (or, equiavently, $\alpha_2 = \alpha_3 = A$, $\beta_2 = B$, $\beta_3 = 2b-1$), then from the equations $\frac{1}{x} = \frac{\sqrt{a-B}}{\sqrt{A-a}}$ and $y = \frac{\sqrt{A-b}}{\sqrt{a}}$ we obtain

$$\alpha_2 = \alpha_3 = A = ay^2 + b, \qquad \beta_2 = B = \frac{a(1+x^2)-(ay^2+b)}{x^2}. \tag{15}$$

The first two equations of (11) take form

$$\begin{cases} qx = py + q, \\ 2ay\left(\frac{q}{x} + p\right) = p(1 - B). \end{cases} \tag{16}$$

From the first equation it follows $x = \frac{py+q}{q}$. Substituting it to the second equation we receive after symplification

$$\left(2\,y^3ap^3 + \left(-p^3 + 4\,ap^2q + paq^2 + ap^3\right)y^2 + \right.$$
$$\left. + \left(2\,aq^3 - 2\,p^2q + 2\,paq^2 + 2\,ap^2q\right)y + 3\,paq^2 - 2\,pq^2\right)/(py+q)^2 = 0. \tag{17}$$

Substituting it to the third equation in (11) we obtain

$$y\left(2\,y^3ap^3 + \left(-p^3 + 4\,ap^2q + paq^2 + ap^3\right)y^2 + \right.$$
$$\left. + \left(2\,aq^3 - 2\,p^2q + 2\,paq^2 + 2\,ap^2q\right)y + 3\,paq^2 - 2\,pq^2\right)/(py+q)^2 = 0.$$

It is suffucient to find only positive roots of (17).

Denoting $\lambda = p/q$ we have

$$2a\lambda^3y^3 + \lambda(a + 4a\lambda - \lambda^2 + a\lambda^2)y^2 + 2(a + a\lambda - \lambda^2 + a\lambda^2)y + \lambda(3a - 2) = 0. \tag{18}$$

Denote the cubic polynomial at the left side of (18) as $\nu(y)$, $\nu(0) = \lambda(3a-2) < 0$, $a \in [0; 0.5)$. The coefficient in higher degree of y in (18) is positive, hence, at least one postive root exists. From here also follows that the maximum lies before minimum. The function $\nu = \nu(y)$ has two extreme points $y_1 = \frac{1}{3}\left(\frac{1}{a} - \frac{1+\lambda+\lambda^2}{\lambda^2}\right)$ and $y_2 = -\frac{1}{\lambda} < 0$. With $\nu(0) < 0$ it gives the uniqueness of the positive root of (18).

The solution takes place in case of $\beta_1 \geq 0$, or $a(3 - y^2) \geq 1$. It determines the lower border $a_2(p)$ of the region D_2 on the plane (p, a).

Theorem 4. For $(p, a) \in D_2$ the equilibrium is (F_1^*, F_2^*) of the form (7–8) with parameters determined by (15–17). The value of the game: $v = q(2a - \beta_2) + p\alpha_3 - p(2b - 1 + \beta_2)\frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}$.

In case $a < a_2(p)$ the following solution will take place.

4.3 Equilibrium for $(p, a) \in D_3$

If $F_1^*(1) < 1$ and $F_2^*(0+) > 0$ (or, equivalently, $\alpha_2 = 2a$, $\beta_3 = 2b - 1$, $\alpha_3 = A$, $\beta_2 = B$), the first two equations in (11) with $1/x = \frac{\sqrt{a-B}}{\sqrt{a}}$ and $y = \frac{\sqrt{A-b}}{\sqrt{a}}$ (or, $\beta_2 = B = a - a/x^2$ and $\alpha_3 = A = ay^2 + b$) take the form

$$\begin{cases} 2a(py + q) = q(ay^2 + b)x, \\ 2ay\left(\dfrac{q}{x} + p\right) = p\left(b + \dfrac{a}{x^2}\right). \end{cases} \tag{19}$$

From the first equation in (19) it follows $x = \dfrac{2a(py+q)}{q(ay^2+b)}$. Substituting it to the second equation in (19) and the third equation in (11) we obtain

$$\left(3\,a^2y^4q^2p + \left(8\,a^2p^3 + 4\,a^2q^3\right)y^3 + \left(-2\,pa^2q^2 - 4\,p^3a + 16\,a^2p^2q + 4\,a^2p^3 + 2\,paq^2\right)y^2 + \right.$$

$$+\left(8\,a^2p^2q+8\,pa^2q^2-4\,a^2q^3-8\,p^2aq+4\,q^3a\right)y+3\,pa^2q^2-pq^2-2\,paq^2)/(4a(py+q)^2)=0. \tag{20}$$

and

$$y(3\,a^2y^4q^2p+\left(8\,a^2p^3+4\,a^2q^3\right)y^3+\left(-2\,pa^2q^2-4\,p^3a+16\,a^2p^2q+4\,a^2p^3+2\,paq^2\right)y^2+$$

$$+\left(8\,a^2p^2q+8\,pa^2q^2-4\,a^2q^3-8\,p^2aq+4\,q^3a\right)y+3\,pa^2q^2-pq^2-2\,paq^2)/(4a(py+q)^2)=0.$$

It is sufficient to find only positive solutions of (20).

Denoting $\lambda = p/q$ we rewrite (20) in the form

$$3a^2\lambda y^4+4a^2(1+2\lambda^3)y^3+2a\lambda(1-a+8a\lambda-2\lambda^2+2a\lambda^2)y^2+$$
$$+4a(1-a+2a\lambda-2\lambda^2+2a\lambda^2)y-(1-a)(1+3a)\lambda=0.$$

Denote $\nu(y)$ polynomials at the left side of the equation. Then $\nu(0)=-(1-a)(1+3a)\lambda<0$, and because the coefficient in higher degree of y is positive then there exists at least one positive root of the equation. Let us show that it is unique. It follows from the fact that the points where $\nu''(y)=0$ are negative.

$$\nu''(y)=36a^2\lambda y^2+24a^2(1+2\lambda^3)y+4a\lambda((1-a)(1-2\lambda^2)+8a\lambda).$$

If this parabola has no roots then $\nu(y)$ is concave and the positive root is unique. Let there are two roots

$$y_{1,2}=\frac{-a(1+2\lambda^3)\pm\sqrt{a(4a\lambda^6-2a\lambda^4+2\lambda^4-4a\lambda^3+a\lambda^2-\lambda^2+a)}}{3a\lambda}.$$

The root y_1 is negative. Coefficient in higher degree of y of $\nu''(y)$ is positive, hence, the largest root y_2 is negative, iff the coefficient in lower degree of $\nu(y)$ is positive. It is equal to $\xi(a,\lambda)=(1-a)(1-2\lambda^2)+8a\lambda$. We have: $\xi(a,0)=1-a>0$, the function $\xi(a,\lambda)$ is convex in λ, $\xi(a,1)=9a-1$. If $a>\frac{1}{9}$, then $\xi(a,\lambda)>0$, coefficient in lower degree in $\nu''(y)$ is positive, y_2 is negative, hence, the positive root of the equation is inique.

The solution takes place, iff $\beta_2\geq 0$ or $\frac{ay^2+b}{2a(1+\lambda y)}\leq 1$. This enequality determines the lower border $a_3(p)$ of the region D_3 on the plane (p,a). Notice, that in D_3 the inequality $a<\frac{1}{9}$ is satisfied automatically.

Theorem 5. For $(p,a)\in D_3$ the equilibrium is (F_1^*,F_2^*) of the form (7–8) with parameters detemined by (19–20). The value of the game: $v=q(2a-\beta_2)+p\alpha_3-p(2b-1+\beta_2)\frac{\sqrt{\alpha_3-b}}{\sqrt{a}}$.

For fixed p, if a decreases from $a_2(p)$ to $a_3(p)$, then β decreases to zero. Finally, consider the case $a<a_3(p)$.

4.4 Equilibrium for $(p,a)\in D_4$

For $\alpha_1=\alpha_2=2a$, $\alpha_4=1$, $\beta_1=\beta_2=0$, $\beta_3=2b-1$ the optimal strategies are

$$F_1^*(s_1)=\begin{cases}0, & \text{if } s_1\leq\alpha_3,\\ \frac{1}{q}\left(1-\frac{\sqrt{\alpha_3-b}}{\sqrt{s_1-b}}\right), & \text{if } \alpha_3<s_1\leq 1,\\ 1, & \text{if } 1<s_1,\end{cases} \tag{21}$$

$$F_2^*(s_2) = \begin{cases} 0, & \text{if } s_2 \le 0, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{a}}, & \text{if } 0 < s_2 \le 2b-1, \\ \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - s_2}}, & \text{if } 2b-1 < s_2 \le 2b - \alpha_3, \\ 1, & \text{if } 2b - \alpha_3 < s_2. \end{cases} \tag{22}$$

Then, for $s_1 \in (\alpha_3; 1]$

$$H_2 = H(s_1, F_2^*(s_2)) = p\left\{0 \cdot F_2^*(0) + \int\limits_{2b-1}^{2b-\alpha_3} s_2 dF_2^*(s_2)\right\} +$$

$$+q\left\{s_1 F_2^*(2b - s_1) + \int\limits_{2b-s_1}^{2b-\alpha_3} s_2 dF_2^*(s_2)\right\} = \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}.$$

If $s_2 = 0$, then

$$H_3 = H(F_1^*(s_1), s_2) = q\left\{\int\limits_{\alpha_3}^{1} s_1 dF_1^*(s_1) + 1 \cdot (1 - F_1^*(1))\right\} = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p.$$

If $s_2 \in (2b-1; 2b-\alpha]$, then

$$H_4 = H(F_1^*(s_1), s_2) = ps_2 + q\left\{\int\limits_{\alpha_3}^{2b-s_2} s_1 dF_1^*(s_1) + s_2(1 - F_1^*(2b - s_2))\right\} = 2b - \alpha_3.$$

$F_1^*(s_1)$, $F_2^*(s_2)$ be optimal iff

$$\begin{cases} 2b - \alpha_3 = \alpha_3 - \frac{p\sqrt{\alpha_3 - b}}{\sqrt{a}}, \\ 2b - \alpha_3 = 2\sqrt{a}\sqrt{\alpha_3 - b} + 2b - \alpha_3 - p. \end{cases}$$

Solution of this system: $\alpha_3 = b + \frac{p^2}{4a}$.

This form for H_2–H_4 takes place, iff $\alpha_3 \le 1$ or, equivalently, $a > p/2$. That determines the region D_4 on the plane (p, a).

Theorem 6. For $(p, a) \in D_4$ the equilibrium is (F_1^*, F_2^*) of the form (21–22). The value of the game: $v = b - \frac{p^2}{4a}$.

The case $a < p/2$ was analysed in section 2.

5 Solution for $p > 0.5$

At the begining we assumed $p \le 0.5$. In case $p > 0.5$ the solution follows from the following theorem.

Theorem 7. Let for some fixed values of a and p we found the optimal strategies $F_1^*(s_1, p, a)$ and $F_2^*(s_2, p, a)$ in the game with

$$P\{\alpha = a\} = p, \quad P\{\alpha = b\} = q, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \le q.$$

Then the optimal strategies in the game for the same values a, p and for

$$P\{\alpha = a\} = q, \quad P\{\alpha = b\} = p, \quad a + b = 1, \quad p + q = 1, \quad a < b, \quad p \le q,$$

are

$$G_1^*(s_1, q, a) = 1 - F_2^*(1 - s_1, p, a), \quad G_2^*(s_2, q, a) = 1 - F_1^*(1 - s_2, p, a).$$

Proof. We have

$$G_1^*(s_1, q, a) = \begin{cases} 0, & \text{if } s_1 \le 1 - 2b + \alpha_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{s_1 - a}}, & \text{if } 1 - 2b + \alpha_3 < s_1 \le 1 - \beta_3, \\ 1 - \frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}}, & \text{if } 1 - \beta_3 < s_1 \le 1 - \beta_2, \\ 1 + \frac{q}{p} - \left(\frac{\sqrt{\alpha_3 - b}}{\sqrt{b - \beta_3}} + \frac{q}{p}\right)\frac{\sqrt{a - \beta_2}}{\sqrt{s_1 - b}}, & \text{if } 1 - \beta_2 < s_1 \le 1 - 2a + \alpha_2, \\ 1, & \text{if } 1 - 2a + \alpha_2 < s_1, \end{cases}$$

$$G_2^*(s_2, q, a) = \begin{cases} 0, & \text{if } s_2 \leqslant 1 - 2b + \beta_3, \\ \left(\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}} + \frac{p}{q}\right)\frac{\sqrt{\alpha_3 - b}}{\sqrt{a - s_2}} - \frac{p}{q}, & \text{if } 1 - 2b + \beta_3 < s_2 \le 1 - \alpha_3, \\ \frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}, & \text{if } 1 - \alpha_3 < s_2 \le 1 - \alpha_2, \\ \frac{\sqrt{a - \beta_2}}{\sqrt{b - s_2}}, & \text{if } 1 - \alpha_2 < s_2 \le 1 - 2a + \beta_2, \\ 1, & \text{if } 1 - 2a + \beta_2 < s_2. \end{cases}$$

These functions will represent the optimal strategies, iff

$$H(s_1, G_2^*(s_2, q, a)) = \text{const for } s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3] \cup (1 - \beta_2; 1 - 2a + \alpha_2],$$

$$H(G_1^*(s_1, q, a), s_2) = \text{const for } s_2 \in (1 - 2b + \beta_3; 1 - \alpha_3] \cup (1 - \alpha_2; 1 - 2a + \beta_2].$$

Denote $G_{1,12}^*(s_1)$ and $G_{1,34}^*(s_1)$ as the form of function $G_1^*(s_1, q, a)$ at the intervals $(1 - 2b + \alpha_3; 1 - \beta_3]$ and $(1 - \beta_2; 1 - 2a + \alpha_2]$ and $G_{2,12}^*(s_1)$, $G_{2,34}^*(s_1)$ for the $G_2^*(s_1, q, a)$ at the intervals $(1 - 2b + \beta_3; 1 - \alpha_3]$, $(1 - \alpha_2; 1 - 2a + \beta_2]$, respectively.

We obtain for $s_1 \in (1 - 2b + \alpha_3; 1 - \beta_3]$

$$H_1' = H(s_1, G_2^*(s_1, q, a)) = q\left\{s_1 G_{2,12}^*(2a - s_1) + \int\limits_{2a - s_1}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int\limits_{1 - \alpha_2}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2)\right\} +$$

$$+ps_1 = q\frac{\sqrt{a - \beta_2}}{\sqrt{\alpha_2 - a}}((\alpha_3 - 2b) - (\alpha_2 - 2a)) + p(\alpha_3 + 2a - 1) + q(1 - \beta_2).$$

If $s_1 \in (1 - \beta_2; 1 - 2a + \alpha_2]$, then

$$H_2' = H(s_1, G_2^*(s_1, q, a)) = q\left\{0 \cdot G_2^*(0, q, a) + \int\limits_{1 - 2b + \beta_3}^{1 - \alpha_3} s_2 dG_{2,12}^*(s_2) + \int\limits_{1 - \alpha_2}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2)\right\} +$$

$$+p\left\{s_1 G_{2,34}^*(2b - s_1) + \int\limits_{2b - s_1}^{1 - 2a + \beta_2} s_2 dG_{2,34}^*(s_2)\right\} =$$

$$= q\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}((\alpha_3-2b)-(\alpha_2-2a))+p(\alpha_3+2a-1)+q(1-\beta_2)-$$

$$-q(1-\beta_3)\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}\cdot\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}+p(1-\beta_2)-p(1-\beta_3)\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}.$$

If $s_2 \in (1-2b+\beta_3; 1-\alpha_3]$, then

$$H_3' = H(G_1^*(s_1,q,a),s_2) = q\left\{\int\limits_{1-2b+\alpha_3}^{2a-s_2} s_1 dG_{1,12}^*(s_1) + s_2(1-G_{1,12}^*(2a-s_2))\right\}+$$

$$+p\left\{\int\limits_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) + \int\limits_{1-\beta_2}^{1-2a+\alpha_2} s_1 dG_{1,34}^*(s_1) + 1\cdot(1-G_1^*(1))\right\} =$$

$$= p\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}((2b-\beta_3)-(2a-\beta_2))+p(1-\alpha_3)-q(1-2b-\beta_2)-$$

$$-p(1-\alpha_2)\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}+q(1-\alpha_3)-q(1-\alpha_2)\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}+p\eta.$$

If $s_2 \in (1-\alpha_2; 1-2a+\beta_2]$, then

$$H_4' = H(G_1^*(s_1,q,a),s_2) = qs_2 + p\left\{\int\limits_{1-2b+\alpha_3}^{1-\beta_3} s_1 dG_{1,12}^*(s_1) +\right.$$

$$\left. + \int\limits_{1-\beta_2}^{2b-s_2} s_1 dG_{1,34}^*(s_1) + s_2(1-G_{1,34}^*(2b-s_2))\right\} =$$

$$= p\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}((2b-\beta_3)-(2a-\beta_2))+p(1-\alpha_3)-q(1-2b-\beta_2),$$

where $\eta = \begin{cases} 0, \text{ if } G_1^*(1)=1, \\ -\frac{q}{p}+\left(\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}+\frac{q}{p}\right)\frac{\sqrt{a-\beta_2}}{\sqrt{a}}, \text{ if } G_1^*(1)<1. \end{cases}$

We have

$$\psi_1 = H_2' - H_1' = -q(1-\beta_3)\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}\cdot\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}+p(1-\beta_2)-p(1-\beta_3)\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}},$$

$$\psi_2 = H_3' - H_4' = -p(1-\alpha_2)\frac{\sqrt{\alpha_3-b}}{\sqrt{b-\beta_3}}\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}+q(1-\alpha_3)-q(1-\alpha_2)\frac{\sqrt{a-\beta_2}}{\sqrt{\alpha_2-a}}+p\eta.$$

There are only four possible forms for the functions $F_1^*(s_1,p,a)$ and $F_2^*(s_2,p,a)$. With $\chi_1=\chi_2=0$, it gives:

1. For $\alpha_2=\alpha_3=A$, $\beta_2=\beta_3=B$ take place $\frac{\chi_1}{A}=\frac{\psi_2}{1-A}$ and $\frac{\chi_2}{B}=\frac{\psi_2}{1-B}$, consequently, $\psi_1=\psi_2=0$.

2. For $\alpha_2=\alpha_3=A$, $\beta_3=2b-1$ take place $\frac{\chi_1}{A}=\frac{\psi_2}{1-A}$ and $\chi_2=-\psi_1$, consequently, $\psi_1=\psi_2=0$.

. For $\alpha_2 = 2a$, $\beta_3 = 1-2a$ take place $\chi_1 = -\psi_2$ and $\chi_2 = -\psi_1$, consequently, $\psi_1 = \psi_2 = 0$.

. For $\alpha_1 = \alpha_2 = 2a$, $\alpha_4 = 1$, $\beta_1 = \beta_2 = 0$, $\beta_3 = 2b-1$, the form of $G_1^*(s_1)$, $G_2^*(s_2)$ is:

$$G_1^*(s_1,q,a) = \begin{cases} 0, & \text{if } s_1 \le a + \dfrac{p^2}{4a}, \\ 1 - \frac{p}{2\sqrt{a}\sqrt{s_1-a}}, & \text{if } a + \frac{p^2}{4a} < s_1 \le 2a, \\ 1 - \frac{p}{2a}, & \text{if } 2a < s_1 \le 1, \\ 1, & \text{if } 1 < s_1, \end{cases}$$

$$G_2^*(s_2,q,a) = \begin{cases} 0, & \text{if } s_2 \le 0, \\ 1 - \frac{1}{q}\left(1 - \frac{p}{2\sqrt{a}\sqrt{a-s_2}}\right), & \text{if } 0 < s_2 \le a - \frac{p^2}{4a}, \\ 1, & \text{if } a - \frac{p^2}{4a} < s_2. \end{cases}$$

Then for $s_2 \in \left(0; a - \frac{p^2}{4a}\right]$

$$H(G_1^*(s_1,q,a), s_2) = q\left\{\int\limits_{a+\frac{p^2}{4a}}^{2a-s_2} s_1 dG_1^*(s_1,q,a) + s_2(1 - G_1^*(2a-s_2,q,a))\right\} +$$

$$+p\left\{\int\limits_{a+\frac{p^2}{4a}}^{2a} s_1 dG_1^*(s_1,q,a) + 1\cdot(1 - G_1^*(1,q,a))\right\} = a + \frac{p^2}{4a}.$$

For $s_1 \in \left(a + \frac{p^2}{4a}; 2a\right]$

$$H(s_1, G_2^*(s_2,q,a)) = q\left\{s_1 G_2^*(2a-s_1,q,a) + \int\limits_{2a-s_1}^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2,q,a)\right\} + ps_1 = a + \frac{p^2}{4a}.$$

Finally, for $s_1 = 1$

$$H(s_1, G_2^*(s_2,q,a)) = q\int\limits_0^{a-\frac{p^2}{4a}} s_2 dG_2^*(s_2,q,a) + p = a + \frac{p^2}{4a}.$$

In all cases the payoff is constant, and with $H_1 + H_4' = 1$, $H_4 + H_1' = 1$ and $H_1 = H_4$, ;ives $H_1' = H_4'$, and all $H_i', i = 1,..,4$ are equal. It proves the optimality $G_1^*(s_1,q,a)$ and (s_2,q,a).

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